16.7 Surface Integrals

Surface Integrals

As a line integral integrates **w.r.t. arc length**, a surface integral integrates **w.r.t. surface area**.

$$A(S) = \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA$$

Surface integrals will **comprise the majority** of the **rest** of the **chapter**! First, consider **surfaces** given by ***z* = *f*(*x*, *y*)**.

Let ***S*** be a surface given by ***z* = *f*(*x*, *y*)** and let ***R*** be its **projection** onto the ***xy*-plane**. Suppose that f, f_x, and f_y are **continuous** at all points in *R* and that *f* is defined on *S*.

Sketch:

Parametric Surfaces

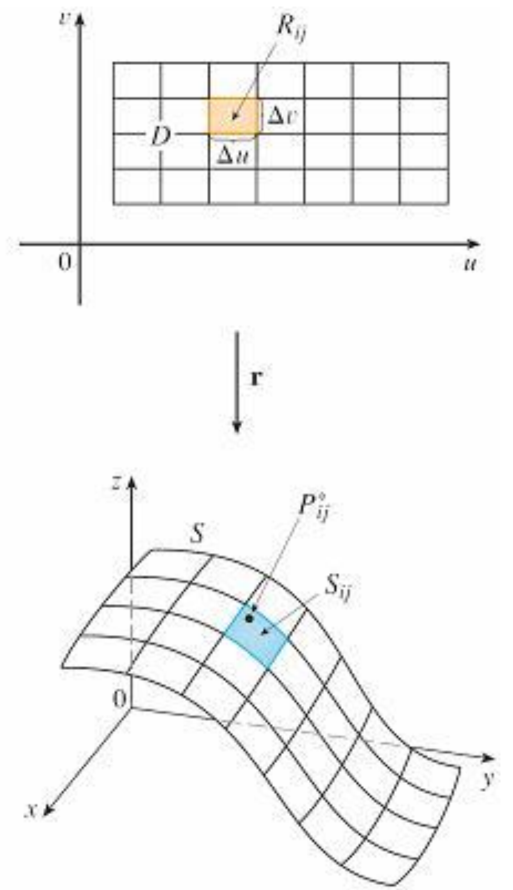
Parametric Surfaces

Let's suppose that a surface *S* has a vector equation

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k} \qquad (u, v) \in D$$

Given a **function in 3-space**, then evaluate it at **some** P_{ij} on each "**patch**" of S and **multiply by the area** ΔS_{ij} of patch, to get Riemann Sum:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^{*})\,\Delta S_{ij}$$



As the number of "patches" **increases**:

$$\iint_S f(x, y, z)\,dS = \lim_{m,\,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^{*})\,\Delta S_{ij}$$

Called the **surface integral of f over S**.

Parametric Surfaces

Moreover, if the components are **continuous** and **$\mathbf{r}_u$ & $\mathbf{r}_v$** are nonzero and nonparallel in the interior of *D*, it can be shown, even when ***D* is not a rectangle**, that

$$\iint_S f(x, y, z)\, dS = \iint_D f(\mathbf{r}(u, v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

Also, note that

$$\iint_S 1\, dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA = A(S)$$

Upon using this formula, recall ***f*(r(*u*, *v*))** is generally evaluated by writing ***x* = *x*(*u*, *v*)**, ***y* = *y*(*u*, *v*)**, and ***z* = *z*(*u*, *v*)** in the formula for ***f*(*x*, *y*, *z*)**.

Example – Computing Surface Area

Evaluate the surface integral.

$$\iint_S xyz\, dS$$

where S is the cone with parametric equations

$$x = u\cos v, \quad y = u\sin v, \quad z = u, \quad 0 \le u \le 1, \quad 0 \le v \le \frac{\pi}{2}$$

Example – Computing Surface Area

Evaluate the surface integral.

$$\iint_S xyz\,dS$$

$\vec{r}_u = \langle \cos v, \sin v, 0 \rangle$

$\vec{r}_v = \langle -u\sin v, u\cos v, 1 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle \sin v, -\cos v, u \rangle$

$|\vec{r}_u \times \vec{r}_v| = \sqrt{1+u^2}$

where S is the cone with parametric equations

$$x = u\cos v,\quad y = u\sin v,\quad z = u,\quad 0 \le u \le 1,\quad 0 \le v \le \frac{\pi}{2}$$

$$\iint_A u\cos v\; u\sin v\; u\sqrt{1+u^2}\,du\,dv = \int_0^{\pi/2} \sin v\cos v \int_0^1 u^3\sqrt{1+u^2}\,du$$

Parameterization of a Surface

Important: **Any** surface $z = g(x, y)$ can be parameterized by

$$\boldsymbol{x = x \qquad y = y \qquad z = g(x, y)}$$

And this leads to

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

Which changes our surface integral to

$$\iint_S f(x, y, z)\, dS = \iint_D f(x, y, g(x, y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

Other Parameterizations

Other ways to approach the surface integral:

$$\iint_S f(x, y, z)\, dS = \iint_D f\big(x, h(x, z), z\big) \sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1}\, dA$$

Example – Computing Surface Area

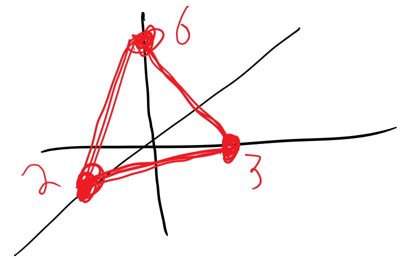
Evaluate the surface integral.

$\iint_S y \, dS$,

S is the part of the plane $3x + 2y + z = 6$ that lies in the first octant

Example – Computing Surface Area

Evaluate the surface integral.



$y = -\frac{3}{2}x + 3$

$\iint_S y\, dS$,

S is the part of the plane $3x + 2y + z = 6$ that lies in the first octant

$z = 6 - 3x - 2y$

$\int_0^2 \int_0^{-3/2x+3} y\sqrt{9+4+1}\; dy\, dx$

Example – Computing Surface Area

Evaluate the surface integral.

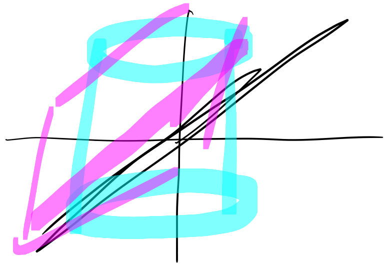
$\iint_S x \, dS,$

S is the surface $y = x^2 + 4z, 0 \leqslant x \leqslant 2, 0 \leqslant z \leqslant 2$

$$\int_0^2 \int_0^2 x \sqrt{(2x)^2 + 16 + 1} \, dx \, dz$$

Example – Computing Surface Area

Evaluate the surface integral.



$\iint_S yz\, dS,$

S is the part of the plane $z = y + 3$ that lies inside the cylinder $x^2 + y^2 = 1$

$\frac{\partial z}{\partial x} = 0 \qquad \frac{\partial z}{\partial y} = 1$

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y(y+3)\sqrt{2}\; dy\, dx$$

$x = r\cos\theta$

$y = r\sin\theta$

$z = r\sin\theta + 3$

$\vec{r}_r = \langle \cos\theta, \sin\theta, \sin\theta \rangle$

$\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, r\cos\theta \rangle$

$x = \ ?$

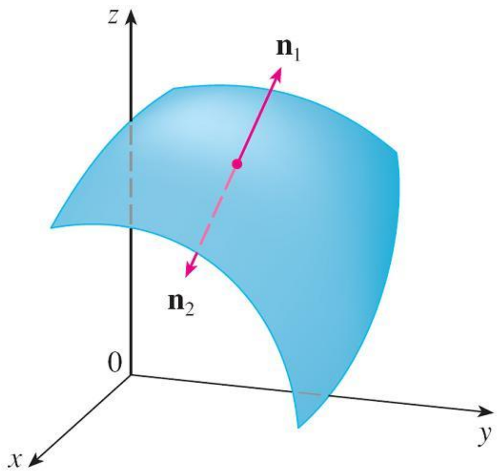
$$\int_0^1\int_0^{2\pi} r\sin\theta(r\sin\theta + 3)$$

Oriented Surfaces

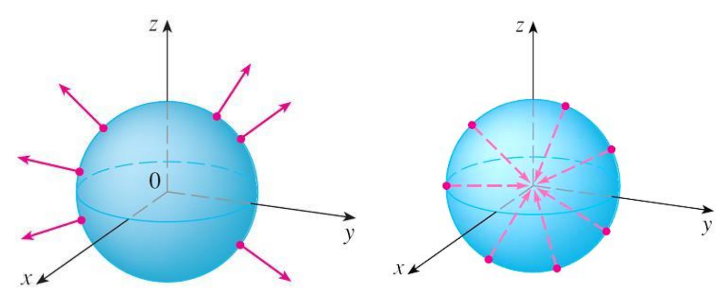
Oriented Surfaces

Let S be an **oriented surface**. S is an "**oriented**" surface only if it is possible to choose a unit normal vector **n** at each point (x, y, z) such that **n varies continuously** over S. In other words, the surface is "**two-sided**."

We start with a surface *S* that has a **tangent plane** at **every point** (*x*, *y*, *z*) on *S* (except at any boundary point). There are **two unit normal** vectors $\mathbf{n}_1$ and $\mathbf{n}_2 = -\mathbf{n}_1$ at (*x*, *y*, *z*).



There are **two possible** orientations for any **orientable** surface, **positive** (or **outward**) and **negative** (or **inward**).



Oriented Surfaces

For $z = g(x, y)$ given as the graph of g associated with the surface with natural orientation given by the **unit normal vector**:

$$\mathbf{n} = \frac{-\dfrac{\partial g}{\partial x}\mathbf{i} - \dfrac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\dfrac{\partial g}{\partial x}\right)^2 + \left(\dfrac{\partial g}{\partial y}\right)^2}}$$

Note: the **k**-component is **positive**, so an ***upward* orientation**.

Rather,

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

Again, this is **positive orientation**, simply apply **–n** for **negative**.

Surface Integrals of Vector Fields

Surface Integrals of Vector Fields

A primary application involving the surface integral relates to the **flow of a fluid through a surface *S***.

Suppose an **oriented** surface *S* is **submerged in a fluid** having a **continuous velocity field F**, the volume of fluid **crossing the surface *S*** per unit of time (called the **flux of F across *S***) is given by the following surface integral:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS$$

where **n** is the **unit normal vector to S**.

The above surface integral measures the **net flow amount** of fluid (**H_2O, air, etc.**) out **over a surface area**.

Surface Integrals of Vector Fields

Note that if S is given by a vector function $\mathbf{r}(u, v)$, then $\mathbf{n}$ is given by

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left[\mathbf{F}(\mathbf{r}(u, v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \right] |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

where D is the parameter domain. Thus we have

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA$$

Example – Computing Flux

Evaluate the surface integral for the given vector field **F** and the oriented surface *S*. In other words, find the flux of **F** across *S*. For closed surfaces, use the positive (outward) orientation.

$$\mathbf{F}(x, y, z) = e^{y}\,\mathbf{i} + ye^{x}\,\mathbf{j} + x^{2}y\,\mathbf{k},$$

S is the part of the paraboloid $z = x^2 + y^2$ that lies above the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$ and has upward orientation

Example – Computing Flux

Evaluate the surface integral for the given vector field **F** and the oriented surface *S*. In other words, find the flux of **F** across *S*. For closed surfaces, use the positive (outward) orientation.

$\mathbf{F}(x, y, z) = x\,\mathbf{i} + xy\,\mathbf{j} + xz\,\mathbf{k}$,

S is the part of the plane $3x + 2y + z = 6$ that lies in the first octant

Example – Computing Flux

Evaluate the surface integral for the given vector field **F** and the oriented surface *S*. In other words, find the flux of **F** across *S*. For closed surfaces, use the positive (outward) orientation.

$$\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k},$$

S is the sphere $x^2 + y^2 + z^2 = 9$

Surface Integrals (Book)

For instance, if **E** is an electric field, then the surface integral

$$\iint_S \mathbf{E} \cdot d\mathbf{S}$$

is called the **electric flux** of **E** through the surface *S*. One of the important laws of electrostatics is **Gauss's Law**, which says that the net charge enclosed by a closed surface *S* is

$$\boxed{11} \qquad Q = \varepsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S}$$

where ε_0 is a constant (called the permittivity of free space) that depends on the units used. (In the SI system, $\varepsilon_0 \approx 8.8542 \times 10^{-12}\ C^2/N \cdot m^2$.)

Surface Integrals (Book)

Therefore, if the vector field **F** in Example 4 represents an electric field, we can conclude that the charge enclosed by *S* is

$$Q = \frac{4}{3}\pi\varepsilon_0.$$

Another application of surface integrals occurs in the study of heat flow. Suppose the temperature at a point (x, y, z) in a body is $u(x, y, z)$. Then the **heat flow** is defined as the vector field

$$\mathbf{F} = -K\,\nabla u$$

where K is an experimentally determined constant called the **conductivity** of the substance.

Lots of applications!!